TIME-DEPENDENT VARIATIONAL APPROACH IN TERMS OF SQUEEZED COHERENT STATES
—IMPLICATION TO SEMI-CLASSICAL APPROXIMATION—

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Abstract

A general framework for time-dependent variational approach in terms of squeezed coherent states is constructed with the aim of describing quantal systems by means of classical mechanics including higher order quantal effects with the aid of canonicity conditions developed in the time-dependent Hartree-Fock theory. The Maslov phase occurring in a semi-classical quantization rule is investigated in this framework. In the limit of a semi-classical approximation in this approach, it is definitely shown that the Maslov phase has a geometric nature analogous to the Berry phase. It is also indicated that this squeezed coherent state approach is a possible way to go beyond the usual WKB approximation.

1 Introduction

In many-body problems, a great interest is paid to describe quantal systems in terms of a few classical variables because we are especially interested in some particular characteristic motions in quantal systems, for example, nuclear collective motions in nucleus and the dynamics of soliton models of baryons as the low energy effective theory of QCD. As is well known, in various quantal systems, if one takes the limit of "large N", the quantum theories are well described as the classical ones [1]. However, since, for example, we are interested in the nuclei as finite quantum many-particle systems, it should be noticed that the deviations from classical dynamics can never be neglected.

With the aim of establishing a possible framework for the classical description of quantal systems, we give a rather general framework to describe quantal systems by means of classical mechanics including the higher order quantal effects. Our basic idea is formulated with the use of the time-dependent variational principle utilizing the squeezed coherent states [2], paying strong attention to canonicity conditions developed in the TDHF theory [3][4].

In this paper, first, we briefly review our time-dependent variational approach with squeezed coherent states developed in Refs.[2] and [5]. Secondly, we show that, when we take a semi-classical limit in our framework, it is clearly realized that the Maslov correction occurring in the semi-classical quantization procedure in the usual WKB method can directly be interpreted as the Berry phase [6]. Although it has originally been pointed out that the Maslov correction is a kind of the Berry phase [7], it is possible to take account of the higher order quantum effects than that of the semi-classical approximation in our framework. Furthermore, it is understood that our approach is a possible way to go beyond the usual WKB approximation.

2 Formulation

In this section, we give the framework of the time-dependent variational approach in terms of squeezed coherent states [5][6]. We start with the general squeezed coherent state as

$$|\Phi(\alpha,\beta)\rangle \equiv \exp\{\sum_k (\alpha_k \hat{a}_k^\dagger - \alpha_k^* \hat{a}_k)\}|\Psi(\beta)\rangle , \tag{1}$$

$$|\Psi(\beta)\rangle \equiv \exp\{\frac{1}{2}\sum_{k,k'}(\hat{a}_k^\dagger B_{kk'}\hat{a}_{k'}^\dagger - \hat{a}_k B_{kk'}^* \hat{a}_{k'})\}|0\rangle . \tag{2}$$

Here, $|0\rangle$ is a vacuum state with respect to boson operators $\hat{a}_k$, and α_k and $B_{kk'}$ are the time-dependent c-number variables. The state $|\Psi(\beta)\rangle$ is called the squeezed vacuum. In the following consideration, we are restricted ourselves to deal with boson systems composed of one kind of boson. If we want to consider the systems described by su(2)-algebra such as the Lipkin model, it is enough to express the algebra by the use of two-kinds of boson operators, the representation of which is well known as Schwinger boson representation. Then, $B_{kk'}$ is taken as $B_{kk'} = B_k \delta_{kk'}(k = 1,2)$ [8].

With the aid of definitions of coordinate-momentum operators $\hat{Q} = \sqrt{\hbar/2}(\hat{a} + \hat{a}^\dagger)$ and $\hat{P} = (-i)\sqrt{\hbar/2}(\hat{a} - \hat{a}^\dagger)$, the above squeezed coherent state can be rewritten as the following Gaussian-type state :

$$\begin{aligned}|\Phi(t)\rangle &\equiv (2G)^{-\frac{1}{4}}\exp\{\frac{i}{\hbar}(p\hat{Q} - q\hat{P})\}\exp\{\frac{1}{2\hbar}\Omega\hat{Q}^2\}|0\rangle \\ &= e^{-i\varphi}|\Phi(\alpha,\beta)\rangle ,\end{aligned} \tag{3}$$

where we define the following variables as

$$q \equiv \sqrt{\frac{\hbar}{2}}(\alpha + \alpha^*) , \qquad p \equiv (-i)\sqrt{\frac{\hbar}{2}}(\alpha - \alpha^*) , \tag{4}$$

$$\Omega = 1 - \frac{1}{2G} + i2\Pi , \tag{5}$$

$$G \equiv \frac{1}{2}\left|\cosh|B| + \frac{B}{|B|}\sinh|B|\right|^2 , \qquad \Pi \equiv \frac{i}{2}\frac{B^* - B}{|B|}\frac{\sinh|B|\cosh|B|}{\left|\cosh|B| + \frac{B}{|B|}\sinh|B|\right|^2} , \tag{6}$$

$$e^{-i2\varphi} \equiv \frac{1}{\sqrt{2G}}\left(\cosh|B| + \frac{B}{|B|}\sinh|B|\right) . \tag{7}$$

Here, c-number variable Ω is divided into real and imaginary parts, and for later convenience, the part "1" which represents the width of the wave packet of the original vacuum is extracted from real part. In the following, we will start with this expression of the squeezed coherent state in Eq.(3). Thus, we treat the variables q, p, G and Π as dynamical ones. Here, note that the variable G is positive definite and never takes zero. This fact is important in order to present an interpretation of the usual WKB approximation within our framework.

In general, we can calculate the expectation values for arbitrary operators in terms of the Wigner transform :

$$\langle\Phi(t)|\hat{\mathcal{O}}|\Phi(t)\rangle = \exp\{\hbar\hat{\mathcal{D}}\}\mathcal{O}_W(q,p) . \tag{8}$$

Here, the derivative operator $\hat{\mathcal{D}}$ and the Wigner transform $O_W(q,p)$ are defined as

$$\hat{\mathcal{D}} \equiv \frac{1}{2}G\Big(\frac{\partial}{\partial q}\Big)^2 + 2G\Pi\Big(\frac{\partial^2}{\partial q\partial p}\Big) + \frac{1}{2}\Big(\frac{1}{4G} + 4G\Pi^2\Big)\Big(\frac{\partial}{\partial p}\Big)^2 , \tag{9}$$

$$O_W(q,p) \equiv \int_{-\infty}^{\infty} ds\, e^{ips/\hbar}\langle q - \frac{s}{2}|\hat{O}|q + \frac{s}{2}\rangle , \tag{10}$$

where the relation $\hat{Q}|q\rangle = q|q\rangle$ is satisfied. The Wigner transform O_W only depends on q and p and the variables G and Π are introduced by the operation of $\hat{\mathcal{D}}$.

We need to determine the time-development of the variables $q(t), p(t), G(t)$ and $\Pi(t)$, so that the time-development of the state $|\Phi(t)\rangle$ is determined. We can carry this out with the aid of the time-dependent variational principle similar to the TDHF theory :

$$\delta \int_{t_1}^{t_2} dt\langle\Phi(t)|i\hbar\frac{\partial}{\partial t} - \hat{H}|\Phi(t)\rangle = 0 . \tag{11}$$

Furthermore, we impose the canonicity conditions developed in the TDHF theory [4] in order to extract canonical variables. Taking the freedom of canonical transformations into account, we can express the canonicity conditions in the following form :

$$\langle\Phi(t)|i\hbar\partial_X|\Phi(t)\rangle = Y + \partial_X s(X,Y) , \qquad \langle\Phi(t)|i\hbar\partial_Y|\Phi(t)\rangle = \partial_Y s(X,Y) , \tag{12}$$

where $\partial_F \equiv \partial/\partial F$ is defined and $s(X,Y)$ which represents the freedom of the canonical transformation is an arbitrary function of canonical variables X and Y. We can take possible solutions of the above canonicity conditions as $(X,Y) = (q,p)$ and $(\hbar G,\Pi)$. Therefore, the resultant equations of motion derived from the time-dependent variational principle are nothing but the canonical equations of motion due to the canonicity conditions :

$$\dot{q} = \frac{\partial H}{\partial p} = e^{\hbar\mathcal{D}}\frac{\partial H_W}{\partial p} ,$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -e^{\hbar\mathcal{D}}\frac{\partial H_W}{\partial q} , \tag{13}$$

$$\hbar\dot{G} = \frac{\partial H}{\partial\Pi} = \hbar e^{\hbar\mathcal{D}}\Big\{2G\Big(\frac{\partial^2}{\partial q\partial p}\Big) + 4G\Pi\Big(\frac{\partial}{\partial p}\Big)^2\Big\}H_W ,$$

$$\hbar\dot{\Pi} = -\frac{\partial H}{\partial G} = -\hbar e^{\hbar\mathcal{D}}\Big\{\frac{1}{2}\Big(\frac{\partial}{\partial q}\Big)^2 + 2\Pi\Big(\frac{\partial^2}{\partial q\partial p}\Big) + \frac{1}{2}\Big(-\frac{1}{4G^2} + 4\Pi^2\Big)\Big(\frac{\partial}{\partial p}\Big)^2\Big\}H_W . \tag{14}$$

Here, the dot denotes the time-derivative and the c-number Hamiltonian function H is defined by $H \equiv \langle\Phi(t)|\hat{H}|\Phi(t)\rangle = e^{\hbar\mathcal{D}}H_W(q,p)$. Thus, our main task is reduced to solving the classical equations of motion under appropriate initial conditions in the canonical form. As is seen from Eqs.(13) and (14), roughly speaking, the variables q and p represent the classical motion and G and Π may be regarded as the classical images of quantum fluctuations.

3 Maslov Phase as Berry Phase

In this section, we give a relation between the usual WKB approximation and our framework of the time-dependent variational approach with squeezed coherent states. Then, it is clearly shown

that the Maslov correction occurring in the semi-classical quantization procedure in the usual WKB method can directly be interpreted as the Berry or geometric phase.

In our framework, it is necessary to choose the initial conditions for newly-introduced variables as the classical image of quantum fluctuations, that is G and Π. We adopt two criteria developed in our papers [2][6], namely the requirements of "Least Quantal Effects" and "Minimal Uncertainty" at initial time. As for the "classical parts" $q(t)$ and $p(t)$, we may select the initial conditions in a similar way to the usual TDHF theory [9].

Now, if the limit of $\hbar \to 0$ is taken in Eq.(13), then these equations are reduced to the usual classical Hamilton's equations of motion. Thus, it is expected that the variables G and Π represent the quantum fluctuations around the above-mentioned classical motions. Therefore, it is realized that the semi-classical limit in our framework is to take the limit of $\hbar \to 0$ in the equations of motion in Eqs.(13) and (14). In this limit, we can solve the equations of motion for G and Π in Eq.(14) and express these solutions in terms of the classical orbit $(q(t), p(t))$. The results are obtained as follows :

$$G = \frac{1}{2}\left[2G_0A^2 + \frac{B^2}{2G_0}\right] , \qquad \Pi = \frac{1}{4G}\left[2G_0AC + \frac{BD}{2G_0}\right] , \qquad (15)$$

where $A \equiv \partial q/\partial q_0$, $B \equiv \partial q/\partial p_0$, $C \equiv \partial p/\partial q_0$ and $D \equiv \partial p/\partial p_0$ are defined and the variables with subscript 0 represent the initial values. Since the variables G and Π are always accompanied with $\hbar$, the expectation value of Hamiltonian should also be taken into account up to the order of $\hbar$ in this semi-classical limit. Namely, as the approximate energy expectation value, we adopt $H \simeq H_{cl}(q,p) + \hbar H_{ql}(q,p,G,\Pi)$.

In the usual WKB considerations, the energy is kept in the classical form which does not include $\hbar$. Therefore, in our framework of the time-dependent variational approach with the squeezed coherent states, $\hbar \int dt H_{ql}$ in the action integral should be combined with the requantized phase factor $\int dt\langle\Phi(t)|i\hbar\partial/\partial t|\Phi(t)\rangle$ in order to compare our treatment with the usual WKB one properly. Thus, action function is written as

$$\begin{aligned}
S &\equiv \int_0^{T_{cl}} dt\langle\Phi(t)|i\hbar\frac{\partial}{\partial t} - \hat{H}|\Phi(t)\rangle = \int_0^{T_{cl}} dt\left\{\frac{1}{2}(p\dot{q} - \dot{p}q) - \hbar\Pi G - H\right\} \\
&\cong \int_0^{T_{cl}} dt\left\{\left[p\dot{q} + \hbar\frac{\dot{A}B - A\dot{B}}{4G}\right] - H_{cl}\right\} \quad (\text{+total time-derivative term}) , \qquad (16)
\end{aligned}$$

where it is assumed that the classical orbit is a periodic one, the period of which is written by T_{cl}. According to the requantization procedure similar to the TDHF theory, we set the modified action integral except for the part of "energy" to integer n times $2\pi\hbar$:

$$\int_0^{T_{cl}} dt\left\{p\dot{q} + \hbar\frac{\dot{A}B - A\dot{B}}{4G}\right\} = 2\pi\hbar n . \qquad n : \text{integer} \qquad (17)$$

We rewrite the above relation as

$$\oint_C pdq = 2\pi\hbar\left(n - \frac{\dot{\Gamma}}{2\pi}\right) , \qquad (18)$$

where Γ is defined and is explicitly calculated with the relation $G \equiv |z|^2/2$:

$$\Gamma \equiv \int_0^{T_{cl}} dt\frac{\dot{A}B - A\dot{B}}{4G}$$

$$= -\frac{1}{2}\mathrm{Im}\oint_C \frac{dz}{z} = -\pi\nu \ . \qquad\qquad \nu : \text{integer} \qquad\qquad (19)$$

The above expression is nothing but a requantization condition in the semi-classical approximation. Here, z (G) never passes through the point of origin $z = 0$ $(G = 0)$ as is previously mentioned. Then, G or z undergoes the time-evolution accompanied with the classical motion $q(t)$ through the variables A and B. The integer ν, which corresponds to the Maslov correction occurring in the usual semi-classical quantization procedure in the WKB method, appears as the winding number around the origin $z = 0$ associated with the classical motion. These situations are analogous to the case encountered for the Berry phase [10][11]. Namely, it is understood that, in our squeezed coherent state approach, the Maslov correction or the Maslov "phase" corresponds to the Berry phase and the classical orbit plays a role of an "external parameter." The coefficient π in the Maslov phase Γ may be interpreted as a half of the solid angle that subtends at the "singular point" $G = 0$ $(z = 0)$. Furthermore, the parameter governing the approximation is $\hbar$, so that $\hbar$ plays a role of an "adiabatic parameter" in the consideration of the Berry phase. Therefore, it is clearly realized in our approach that the Maslov correction has the similar geometric aspect to the Berry phase. It is thus shown that the quantum effects are automatically contained in the semi-classical limit in our squeezed coherent states approach.

4 Beyond the WKB Approximation

In the usual WKB method, the energy of the system is kept in the classical form which does not include $\hbar$ and the quantum effects are taken into account only through the requantization condition. On the other hand, in our time-dependent variational approach with the squeezed coherent states, the energy is the expectation value of the Hamiltonian with respect to the squeezed coherent state itself, that is $H = \langle\Phi(t)|\hat{H}|\Phi(t)\rangle$, so that the higher order quantum effects of $\hbar$ are already included. Thus, under the conception of our squeezed coherent states approach, the

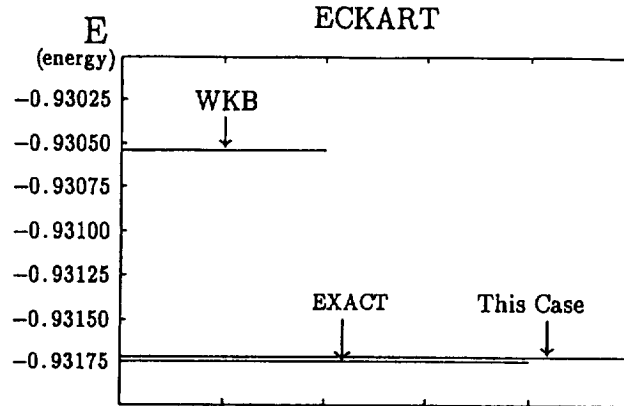

FIG. 1. The energies are shown in the case of Eckart potential $V(Q) = -U_0/\cosh^2\alpha Q$, in which we set the parameters $U_0 = 1$ and $\alpha = 0.1$ for simplicity. "This Case" represents the energy calculated numerically in our squeezed coherent state approach. "WKB" and "Exact" represent the energies obtained by the usual WKB approximation and exact eigenvalue of the ground state, respectively.

energy is calculated as follows : First, we analytically or numerically solve the self-consistent equations of motion in Eqs.(13) and (14). Secondly, we calculate the energy expectation value of the Hamiltonian with respect to the squeezed coherent state which includes the higher order effects of $\hbar$ than the quantum effects in the WKB approximation.

For example, in the case of Eckart potential, $V(\hat{Q}) = -U_0/\cosh^2 \alpha\hat{Q}$, the energy expectation value calculated numerically in our framework is compared with the exact energy eigenvalue and the usual WKB energy in Fig.1. Here, the initial conditions in our approach are taken as $q_0 = p_0 = 0$. Therefore, the energy thus obtained corresponds to the ground state one. It can be seen from Fig.1 that our treatment gives a fairly good result owing to the incorporation of the higher order effects of $\hbar$.

In summary, we have given the framework of the time-dependent variational approach in terms of the squeezed coherent states with the aim of describing quantal systems by means of the classical dynamics. In our squeezed coherent states approach, the Maslov correction that appears in the usual semi-classical quantization procedure is clearly realized as the Berry or geometric phase. Furthermore, our approach is a possible way to go beyond the WKB approximation.

5   Acknowledgments

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